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Exploring Deductive Reasoning Patterns in Proving Logical Statements Using Equivalence Theorem

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Abstract
Deductive reasoning is central to mathematical proof, yet university students frequently struggle to transform logical statements through equivalence theorems. This study analyzed mathematics education students' deductive reasoning abilities and explained the reasoning processes underlying high, moderate, and low performance in formal logical proof tasks. An explanatory sequential mixed-methods design was employed. Quantitative data were collected from 46 mathematics education students through five open-ended deductive reasoning test items, while qualitative data were obtained from six purposively selected students through semi-structured interviews. Quantitative data were analyzed descriptively, and qualitative data were analyzed through data condensation, data display, and conclusion drawing. The results showed that students' overall deductive reasoning performance was relatively low ($M = 56.56$; $SD = 19.83$), with 60.87% of students categorized as low performers. Students performed better in identifying propositional forms (68%) than in selecting equivalence theorems (54%), maintaining logical transformation consistency (49%), and producing valid conclusions (44%). Qualitative analysis revealed three reasoning patterns: strategic-reflective reasoning among high performers, partially developed but inconsistent reasoning among moderate performers, and procedural-fragmented reasoning among low performers. These findings indicate that successful logical proof construction depends not only on knowledge of equivalence rules but also on theorem selection, transformation monitoring, and metacognitive regulation. The study contributes to mathematics education by clarifying the cognitive and metacognitive characteristics of deductive reasoning in formal logic tasks and by offering implications for proof-oriented logic instruction.

Keywords: Deductive Reasoning; Equivalence Theorem; Logical Proof; Mathematical Logic; Mixed Methods; Proof Construction

INTRODUCTION

Mathematical proof is a central epistemic practice in mathematics because it establishes the validity of claims, communicates mathematical relationships, and develops disciplined forms of reasoning. Recent reviews of proof and proving research show that proof remains a major focus in mathematics education because it connects the content of mathematics with students' ways of justifying, explaining, and validating mathematical ideas (Stylianides et al., 2024). In undergraduate mathematics, proof is even more consequential because it functions not only as a learning goal but also as a dominant mode through which mathematical knowledge is presented, assessed, and communicated (Mejia-Ramos et al., 2017; Melhuish et al., 2022).

Within this context, deductive reasoning is an indispensable component of proof construction. Deductive reasoning enables students to derive a conclusion from premises through valid inferential steps and to preserve truth through logically justified transformations. In formal logic courses, this competence becomes highly visible in tasks that require students to prove logical equivalences, because students must identify the structure of compound propositions, select appropriate equivalence laws, transform statements step by step, and justify why the final statement follows from the initial proposition (Chipowe & Hachikona, 2018; Selden & Selden, 2008; Weber, 2001).

Research has repeatedly shown that students' difficulties in proof construction cannot be explained simply as a lack of knowledge of definitions or theorems. Weber (2001) argued that undergraduates may know relevant facts but still fail to produce valid proofs because they lack strategic knowledge about when and how such facts should be used. More recent empirical work has strengthened this view by conceptualizing mathematical proof skill as a resource-based cognitive skill involving topic knowledge, methodological knowledge, strategic knowledge, problem-solving resources, and metacognitive control (Sommerhoff et al., 2021, 2025). Thus, proof construction requires the coordination of several cognitive resources rather than the mechanical recall of proof rules.

The distinction between proof construction, proof comprehension, and proof validation further underscores the need to examine students' reasoning processes during proof writing. Neuhaus-Eckhardt et al. (2025), based on data from 153 undergraduate mathematics students, found that proof construction is empirically separable from proof comprehension and validation, although the skills are strongly related. This finding suggests that students who can recognize a valid proof or understand isolated proof steps may still experience difficulty generating a coherent proof independently. Therefore, studies that focus only on final written products may miss important reasoning processes that occur while students select strategies, monitor transformations, and evaluate conclusions.

Proofs using equivalence theorems in propositional logic represent a particularly demanding context for investigating deductive reasoning. Unlike routine symbolic manipulation, equivalence-theorem proof requires each transformation to preserve logical equivalence and to move purposefully toward a target form. Students must therefore coordinate symbolic syntax, semantic meaning, rule selection, and conclusion validity. Recent work on guided logic problems indicates that students often need explicit support to construct and communicate long sequences of logical arguments (Tithi et al., 2025), while broader research on undergraduate proof learning shows that proof tasks become difficult when students are expected to manage multiple resources simultaneously (Sommerhoff et al., 2021).

Evidence from the Indonesian and regional mathematics education contexts shows similar challenges. Teoh et al. (2025), in a mixed-methods study involving 384 Indonesian Grade 10 students, reported that students struggled to establish relationships between premises and conclusions in reasoning-and-proof tasks. Wulan et al. (2021) identified metacognitive failures among Indonesian preservice mathematics teachers when constructing proofs, including failure to detect errors, misdirection in strategy use, and irrelevant proof steps. Recent studies with prospective mathematics teachers also show that proof difficulties often arise from interconnected conceptual and procedural obstacles (Maifa et al., 2025) and that instructional interventions must explicitly support proof-writing and logical reasoning rather than assume that they develop automatically (Nurlaelah et al., 2025).

From a theoretical perspective, deductive reasoning in logical proof construction involves at least four interrelated processes: identifying propositional forms, selecting valid equivalence theorems, maintaining consistency across transformations, and formulating a valid conclusion. These processes correspond to both cognitive and metacognitive dimensions of mathematical reasoning. Cognitive resources allow students to recognize logical structures and apply equivalence laws, whereas metacognitive resources allow them to plan, monitor, and evaluate the adequacy of each proof step (Schoenfeld, 1985; Veenman & van Cleef, 2019). In this sense, proof construction is not only a matter of knowing logical equivalences but also a matter of regulating their use within a purposeful deductive pathway.

Despite the growing body of research on proof and proving, relatively limited attention has been given to university students' deductive reasoning patterns in proving logical statements specifically through equivalence theorems. Many studies have examined proof errors, proof comprehension, proof validation, or proof construction in algebra, geometry, and analysis (Maifa et al., 2025; Mejia-Ramos et al., 2017; Neuhaus-Eckhardt et al., 2025; Weber, 2001). However, fewer studies have integrated quantitative profiles of deductive reasoning performance with qualitative explanations of how students at different performance levels select equivalence rules, maintain transformation coherence, and monitor their reasoning. Addressing this gap is important because equivalence-theorem proof is foundational for mathematical logic and for prospective mathematics teachers' capacity to explain formal reasoning to future learners.

Accordingly, this study aims to analyze university students' deductive reasoning ability in proving logical statements using equivalence theorems and to explore the reasoning processes underlying different levels of deductive reasoning performance. The study was guided by two research questions: (1) What is the profile of university students' deductive reasoning ability in proving logical statements using equivalence theorems? (2) How do students' deductive reasoning processes differ across high, moderate, and low performance categories? By combining descriptive quantitative analysis and qualitative exploration, this study contributes to the literature by identifying performance-based deductive reasoning patterns in logical proof construction and by offering pedagogical implications for teaching mathematical logic in higher education.

METHODS

Research Design

This study employed an explanatory sequential mixed-methods design. In this design, quantitative data are collected and analyzed first, and qualitative data are subsequently used to explain, elaborate, or contextualize the quantitative findings. The design was appropriate because the study did not only seek to describe students' deductive reasoning scores but also to interpret the reasoning processes that produced those scores.

The quantitative phase generated a profile of students' deductive reasoning performance, including score distribution and achievement on each reasoning indicator. The qualitative phase then explored how selected students interpreted proof tasks, selected equivalence theorems, constructed transformations, justified proof steps, and monitored their reasoning. Integration was conducted at the interpretation stage by connecting the quantitative indicators with qualitative reasoning patterns.

Participants and Research Context

The participants were 46 students from the Mathematics Education Study Program at a state Islamic university in North Sumatra, Indonesia. Participants were selected purposively because they had completed coursework related to mathematical logic and had been introduced to logical proof using equivalence theorems. The purposive approach was suitable because the study required participants who had prior instructional exposure to the target proof domain.

After the quantitative analysis, six students were selected for the qualitative phase using maximum variation sampling. Two students were selected from each performance category: high, moderate, and low. This sampling strategy allowed the study to capture variation in reasoning processes and to compare proof construction characteristics across different levels of deductive reasoning performance.

Instruments

Two instruments were used: a deductive reasoning test and a semi-structured interview guide. The deductive reasoning test consisted of five open-ended proof problems requiring students to prove logical statements using equivalence theorems. The problems were designed to assess four indicators of deductive reasoning: identification of propositional forms, selection of equivalence theorems, consistency of logical transformations, and validity of conclusions. The test was developed in accordance with mathematical logic learning outcomes and was reviewed through expert judgment to ensure content relevance and alignment with the intended constructs.

The semi-structured interview guide was used to explore students' reasoning processes. Interview prompts focused on how students interpreted logical statements, why they selected particular equivalence rules, how they justified each transformation step, how they checked the validity of their proof, and what difficulties they experienced when attempting to reach the target conclusion.

Table 1. Deductive Reasoning Indicators and Analytical Focus

Indicator	Analytical focus in written responses	Analytical focus in interviews
Identification of propositional forms	Accuracy in recognizing compound propositions, connectives, and target proof structure.	How students interpreted the given statement and proof objective.
Selection of equivalence theorems	Relevance and correctness of equivalence rules chosen for transformation.	Why students selected particular rules and whether the choice was strategic or random.
Consistency of logical transformations	Continuity of equivalent statements across proof steps without invalid leaps.	How students monitored whether each step preserved logical equivalence.
Validity of conclusions	Alignment between the final statement and the intended proof target.	How students evaluated whether the proof had been completed validly.

Data Collection Procedure

Data collection was conducted in two sequential phases. First, the deductive reasoning test was administered to all 46 participants. Students' written responses were scored using the four deductive reasoning indicators. The scores were then converted to a 0-100 scale and classified into high, moderate, and low categories.

Second, six students representing the three performance categories were interviewed individually. The interviews were conducted after the quantitative analysis so that interview questions could focus on explaining the observed performance patterns. During the interviews, students were asked to reflect on their written proof attempts, explain their reasoning decisions, and describe difficulties encountered during proof construction.

Data Analysis

Quantitative data were analyzed using descriptive statistics, including mean, standard deviation, minimum score, maximum score, frequency, and percentage. Students' performance was classified into three categories: high (scores >= 80), moderate (scores 60-79), and low (scores <= 59). Achievement percentages were also calculated for each deductive reasoning indicator to identify which aspects of proof construction were most challenging.

Qualitative data were analyzed using the interactive model of Miles, Huberman, and Saldana, consisting of data condensation, data display, and conclusion drawing. Data condensation involved selecting interview segments relevant to proof interpretation, theorem selection, transformation consistency, conclusion validation, and metacognitive monitoring. Data display was conducted through category matrices comparing high-, moderate-, and low-performing students. Conclusions were drawn by identifying recurring reasoning patterns and connecting them with the quantitative results.

Mixed-methods integration was conducted through explanatory connection. Quantitative findings identified the main areas of difficulty, while qualitative findings explained how and why those difficulties occurred. The integration focused on whether students' reasoning patterns supported the score distribution and the indicator-level achievement profile.

Ethical and Reporting Considerations

Participants' identities were treated confidentially, and qualitative reporting used performance categories rather than personal names. For journal submission, the authors should insert the institutional ethics approval number or formal waiver statement if required by the target journal. Because the source manuscript did not report an inter-rater reliability coefficient, this issue is acknowledged in the limitations and should be addressed in future studies.

RESULT AND DISCUSSION

Result

Overall Profile of Deductive Reasoning Ability

The deductive reasoning test was administered to 46 mathematics education students. Students' written responses were assessed using four indicators: identification of propositional forms, selection of equivalence theorems, consistency of logical transformations, and validity of conclusions. The descriptive statistics are presented in Table 2.

Table 2. Descriptive Statistics of Students' Deductive Reasoning Scores

Component	Value
Number of students	46
Maximum possible score	100
Minimum possible score	0
Highest score	90
Lowest score	20
Mean score	56.56
Standard deviation	19.83

Table 2 shows that the mean score was 56.56 with a standard deviation of 19.83. This result indicates that students' overall deductive reasoning performance in proving logical statements using equivalence theorems was relatively low. The score range from 20 to 90 also indicates substantial variation among students. A small number of students were able to construct logical proofs with considerable accuracy, but many students still experienced difficulty in applying equivalence principles deductively and consistently.

Table 3. Distribution of Students by Deductive Reasoning Performance Category

Score interval	Category	Number of students	Percentage
>= 80	High	8	17.39%
60-79	Moderate	10	21.74%
<= 59	Low	28	60.87%
Total	-	46	100%

As shown in Table 3, most students were in the low category. Of the 46 students, 28 students (60.87%) were classified as low performers, 10 students (21.74%) were moderate performers, and only 8 students (17.39%) achieved the high category. This distribution suggests that deductive proof using equivalence theorems remained a significant challenge for the majority of students.

Achievement by Deductive Reasoning Indicator

Further analysis was conducted to identify which components of deductive reasoning created the greatest difficulty. The results are presented in Table 4.

Table 4. Achievement Percentages for Each Deductive Reasoning Indicator

No.	Indicator	Success rate	Category
1	Identification of propositional forms	68%	Medium
2	Selection of equivalence theorems	54%	Low
3	Consistency of logical transformations	49%	Low
4	Validity of conclusions	44%	Low

The highest achievement was found in identifying propositional forms (68%), which was categorized as medium. However, performance declined in the subsequent indicators: selection of equivalence theorems (54%), consistency of logical transformations (49%), and validity of conclusions (44%). This pattern indicates that students were more capable of recognizing the initial structure of logical propositions than of using that recognition to construct coherent deductive proof sequences. The lowest achievement in conclusion validity suggests that many students failed to connect their transformation steps to the intended proof target.

Common Error Patterns in Written Proofs

Analysis of students' written responses showed that errors occurred across the proof construction process, not only in the final answer. The common error types are summarized in Table 5.

Table 5. Common Types of Errors in Students' Logical Proofs

No.	Error type
1	Selection of irrelevant premises
2	Omission of necessary reasoning steps
3	Inappropriate application of equivalence principles
4	Failure to reach the intended conclusion

The error patterns in Table 5 confirm that students' difficulties were located throughout the reasoning process. Some students selected premises that were not relevant to the proof objective. Others skipped intermediate steps, causing the proof chain to become incomplete. Several students applied equivalence principles in ways that did not preserve logical equivalence. In many cases, these errors prevented students from reaching a valid conclusion. These findings explain why the indicators related to theorem selection, transformation consistency, and conclusion validity received lower achievement percentages than the identification of propositional forms.

Qualitative Reasoning Patterns Across Performance Levels

The qualitative phase was conducted to explain the quantitative findings in greater depth. Six students were interviewed: two from the high category, two from the moderate category, and two from the low category. The analysis focused on how students understood proof problems, selected equivalence rules, constructed logical transformations, and evaluated their conclusions.

High-performing students demonstrated strategic and reflective reasoning. They generally began by identifying the structure of the proposition and clarifying the target statement. Before writing the proof, they considered possible transformation pathways and selected equivalence theorems based on their relevance to the target form. Their proof steps were more coherent, and they showed stronger awareness of whether each transformation preserved equivalence. These students also engaged in metacognitive monitoring by checking their steps and reconsidering alternative pathways when they encountered uncertainty. However, even high-performing students sometimes produced longer proof sequences than necessary, suggesting that efficiency in proof construction still required development.

Moderate-performing students showed partially developed but inconsistent reasoning. They could usually recognize the general structure of the problem and identify some relevant equivalence rules. However, they often started transforming statements without establishing a clear proof pathway. Consequently, their reasoning shifted between correct and incorrect steps. They sometimes selected appropriate equivalence theorems but were not always accurate in deciding when and how to apply them. Their monitoring was limited, and transformation errors were often left uncorrected.

Low-performing students relied primarily on procedural and fragmented reasoning. They experienced difficulty identifying the structure of compound propositions and determining the proof objective. Their selection of equivalence theorems tended to be random rather than strategic. Some attempted to imitate memorized examples without understanding why a particular transformation was valid. Their proof sequences frequently failed to preserve equivalence and often ended before reaching the target conclusion. Evidence of self-evaluation or revision during proof construction was minimal.

Table 6. Integrated Qualitative Explanation of Deductive Reasoning Patterns

Performance category	Dominant reasoning pattern	Theorem selection	Transformation monitoring	Explanatory link to quantitative results
High	Strategic-reflective reasoning	Rules selected according to the structure of the target statement.	Students checked whether each step preserved equivalence and revised when necessary.	Explains high scores in theorem selection, transformation consistency, and conclusion validity.
Moderate	Partially developed but inconsistent reasoning	Some relevant rules were identified, but selection was not always connected to a planned proof pathway.	Monitoring was intermittent; incorrect transformations were not always detected.	Explains moderate identification ability but weaker performance in later proof indicators.
Low	Procedural-fragmented reasoning	Rules were selected randomly or by imitating memorized examples.	Minimal checking of equivalence; proof steps often became disconnected.	Explains the dominance of low achievement in theorem selection, transformation consistency, and conclusion validity.

The integrated analysis in Table 6 shows that performance differences were not merely the result of knowing or not knowing equivalence rules. Rather, the differences were strongly associated with how students planned proof strategies, selected theorem sequences, checked transformation validity, and monitored their progress toward the conclusion. Thus, the qualitative findings explain the quantitative pattern: students' lower achievement in theorem selection, transformation consistency, and conclusion validity was closely related to weak strategic reasoning and limited metacognitive regulation.

Discussions

The findings show that students' deductive reasoning ability in proving logical statements using equivalence theorems was still limited. Although students achieved a moderate success rate in identifying propositional forms, their performance declined substantially in selecting equivalence theorems, maintaining transformation consistency, and formulating valid conclusions. This pattern supports the argument that proof construction is not identical to recognizing mathematical structures or understanding isolated proof elements; it requires the ability to generate and regulate a coherent chain of reasoning (Mejia-Ramos et al., 2017; Neuhaus-Eckhardt et al., 2025; Weber, 2001).

The relatively higher achievement in identifying propositional forms suggests that many students possessed basic declarative knowledge of logical statements. However, the lower scores in theorem selection and transformation consistency indicate that such declarative knowledge was not automatically converted into strategic proof performance. This result is consistent with Weber's (2001) claim that students may know relevant mathematical facts but fail to use them strategically. It also aligns with the resource-based view of proof competence, which emphasizes that successful proving depends on the coordination of topic knowledge, methodological knowledge, strategic knowledge, and problem-solving resources (Sommerhoff et al., 2021, 2025).

The qualitative results deepen this interpretation by showing that high-performing students demonstrated strategic-reflective reasoning. They identified the proof goal, selected equivalence rules based on the target structure, and monitored whether each transformation preserved equivalence. This reasoning pattern reflects the kind of disciplinary activity described in studies of authentic mathematical proof, in which students are expected to use mathematical tools with agency, accuracy, and alignment to disciplinary norms (Melhuish et al., 2022; Stylianides et al., 2024). It also confirms that metacognitive monitoring is essential because students must evaluate the validity of each step rather than merely proceed through memorized transformations (Schoenfeld, 1985; Veenman & van Cleef, 2019).

Moderate-performing students represented a transitional reasoning profile. They could identify some relevant proposition structures and equivalence rules, but they often began transforming statements without a planned pathway. Consequently, their proof chains alternated between valid and invalid steps. This pattern suggests partial coordination between conceptual understanding and strategic control. Similar difficulties have been reported in studies showing that students can understand local parts of a proof while failing to organize those parts into a globally coherent argument (Duval, 2006; Selden & Selden, 2008; Sommerhoff & Ufer, 2019).

Low-performing students displayed procedural-fragmented reasoning. Their theorem selection was often random, their transformations did not consistently preserve equivalence, and their conclusions were frequently incomplete or invalid. These findings are consistent with Indonesian studies showing that proof difficulties among preservice mathematics teachers often involve conceptual obstacles, procedural errors, and weak metacognitive control (Maifa et al., 2025; Wulan et al., 2021). The present study extends those findings to the specific domain of logical proof using equivalence theorems, where the demand for step-by-step transformation consistency is especially strong.

The comparison with prior studies suggests that the main difficulty in this study was not merely mathematical logic content but the integration of logical content with strategic and metacognitive regulation. Teoh et al. (2025) reported that Indonesian students struggled to connect premises and conclusions in reasoning-and-proof tasks, while Nurlaelah et al. (2025) showed that structured support can improve prospective teachers' logical reasoning and proof-writing abilities. The present findings are in line with both studies: students need more than exposure to proof rules; they need instructional experiences that help them plan proof routes, justify rule choices, detect invalid transformations, and revise reasoning when necessary.

The study's main contribution is the identification of three deductive reasoning patterns in equivalence-theorem proof construction: strategic-reflective, partially strategic-inconsistent, and procedural-fragmented. This typology advances the analysis beyond listing errors in students' final written proofs. It explains how different students approach the same proof demand and why similar conceptual knowledge may result in different proof outcomes. In this respect, the study responds to recent calls in proof research to examine students' engagement with proof as a dynamic reasoning process involving students, mathematical content, and instructional demands (Melhuish et al., 2022; Stylianides et al., 2024).

Pedagogically, the findings imply that instruction in mathematical logic should make theorem selection and transformation monitoring explicit. Students should be asked not only to apply equivalence laws but also to explain why a particular law is appropriate at a given step, compare alternative transformation pathways, mark whether each step preserves equivalence, and check whether the final statement matches the target conclusion. Such tasks are consistent with resource-based instructional approaches to proof learning, in which students are supported to integrate conceptual knowledge, methodological knowledge, strategic reasoning, and metacognitive regulation within proof construction tasks (Sommerhoff et al., 2021, 2025).

This study has several limitations. The participants came from one mathematics education program, so generalization to other institutional contexts should be made cautiously. The quantitative analysis was descriptive, and the qualitative phase involved six selected students; therefore, future studies could use larger multi-institutional samples, think-aloud protocols, or intervention designs to test how specific forms of scaffolding improve equivalence-theorem proof construction. Future research may also compare deductive reasoning patterns across different domains of proof, such as algebra, geometry, real analysis, and discrete mathematics, to determine whether the three patterns identified in this study are domain-specific or transferable across proof contexts.

CONCLUSIONS

This study investigated university students' deductive reasoning in proving logical statements using equivalence theorems through an explanatory sequential mixed-methods design. The quantitative findings showed that most students were in the low deductive reasoning category, with the greatest difficulties occurring in theorem selection, transformation consistency, and conclusion validity. The qualitative findings explained these results by revealing three reasoning patterns: strategic-reflective reasoning among high-performing students, partially developed but inconsistent reasoning among moderate-performing students, and procedural-fragmented reasoning among low-performing students. These findings demonstrate that successful logical proof construction depends not only on conceptual knowledge of equivalence rules but also on strategic planning, theorem selection, transformation monitoring, and metacognitive regulation. The study contributes to mathematics education by clarifying how students reason during formal logic proof tasks and by suggesting that instruction should explicitly support proof planning, justification, and self-monitoring. Future research should examine these reasoning patterns across broader mathematical domains and evaluate intervention models that strengthen students' deductive proof construction skills.

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